Triangular Arrays using context-free grammar

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Abstract

In this work, the Hao grammar $G = \{\, u \to u^{b_1+b_2+1}v^{a_1+a_2}, \quad v \to u^{b_2}v^{a_2+1}\,\}$, together with the correspondence between grammars and combinatorial differential equations, is employed to obtain an interpretation of any triangular array of the form

$$T(n,k) = (a_2n + a_1k + a_0)\,T(n-1,k) + (b_2n + b_1k + b_0)\,T(n-1,k-1).$$

This lead to have an interpretation of $T(n,k)$ as an increasing tree. Explicit formulas and structural properties are then derived through analytic differential equations. In particular, the r-Whitney-Eulerian numbers and the cases where $b_2n+b_1k+b_0 = 1$ are obtained explicitly. Applications include new interpretation formulas for the r-Eulerian numbers with generating functions. We also obtain full generating functions for the case $a_2 = -a_1$ using this approach.
Keywords: triangular recurrence, formal grammar, differential equations, r-Eulerian, combinatorial interpretation, r-Whitney–Eulerian.

Introduction

In mathematics, certain sequences of numbers satisfy a triangular recurrence of the form

$$T(n,k) = (a_0 + a_1k + a_2n)T(n-1,k) + (b_0 + b_1k + b_2n)T(n-1,k-1). \tag{1}$$

After discussing the binomial coefficients, the Stirling numbers, and the Eulerian numbers, Graham et all proposed a generalization problem of the form (1) in [22]. In combinatorics, several approaches have been developed regarding these kinds of numbers: the GKP numbers[1].

[1]Graham, Knuth, and Patashnik.

One of the results is due to Neuwirth [19][2], who obtained an explicit formula for the case $b_2 = 0$ using the Galton triangle. Spivey [20] found several cases using finite differences. Analytical approaches have also appeared in various works: Théorêt [2], Wilf [21], Barbero [23].

Grammatical approaches and interpretations have also already been developed by Hao and al. [1], Zhou and al. [3]. To study the real-rootedness of polynomial $T_n(x) = \sum_k T(n,k)x^k$, Hao ([1]) introduced the grammar $G = \{\, u \to u^{b_1+b_2+1}v^{a_1+a_2}, \quad v \to u^{b_2}v^{a_2+1} \,\}$ and proved that

$$\mathcal{G}^n(u^{b_0+b_1+b_2}v^{a_0+a_2}) = \sum_{k=0}^{n} T(n,k)u^{b_2n+b_1k+b_0+b_1+b_2}v^{a_2n+a_1k+a_0+a_2}.$$

On another hand, the context-free grammar, by Chen ([9]), developed by Dumont ([8]) was continued by Randrianirina ([16]) with more species interpretation. These last authors showed the relations between the grammar relation and a system of differential equations. This approach say that we can associate the grammar with a system of differential equations

$$\begin{cases} U' = U^{b_1+b_2+1}V^{a_1+a_2}, & U(0) = u \\ V' = U^{b_2}V^{a_2+1} & V(0) = v; \end{cases}$$

and the solution of this system verifies

$$\begin{cases} U(t) = Gen(u,t) = \sum_{n\geq 0} \mathcal{G}^n(u)\frac{t^n}{n!} \\ V(t) = Gen(v,t) = \sum_{n\geq 0} \mathcal{G}^n(v)\frac{t^n}{n!}. \end{cases}$$

In this work, we exploit and combine the grammar of Hao with this fact to study the sequence of triangular recurrence. The recurrence (1) generally arises in the enumeration of combinatorial structures and objects. Moreover, the associated grammar can provide new combinatorial interpretations of the numbers $\Big(T(n,k)\Big)_{n,k\geq 0}$, as Ramírez [4] did for the r-Whitney numbers.

We consider an initial conditions usually taken as

$$T(0,0) = 1, \quad T(n,k) = 0 \text{ if } n < \max(k,0). \tag{2}$$

We remain in the setting where $a_0 + a_1k + a_2n \neq 0$ and $b_0 + b_1k + b_2n \neq 0$. It is more convenient for us to work in the case where $b_0 + b_1k + b_2n = 1$. However, generalizations of Eulerian numbers where $b_0 + b_1k + b_2n \neq 1$ are also important examples when discussing sequences satisfying (1).

This task begins by recalling the background needed to understand the resolution of combinatorial systems of differential equations. We then present the method we will use, which combines context-free grammars with systems of differential equations. Finally, we conclude with several applications.

1 Recalls

In this section, we recall the notions required for understanding this work. These include the solution of combinatorial differential equations. A complete and detailed account of species theories may be found in Bergeron and all [10]. The methods for solving combinatorial

[2]It is what Spivey([20] said. We check this paper: it is not there, maybe in another literature).

differential equations and systems of combinatorial differential equations are presented by Leroux and Viennot in [14] (see also Bergeron and all [10] Chapter 5). Further developments on $\mathbb{L}$-species, mixed species, and the analysis of initial conditions different from 0 and 1 are given by Randrianirina in [15] (see also Randrianirina [16]).

According to André Joyal ([11], definition 1), a species of structure, also known as a $\mathbb{B}$-species, is a functor from the category $\mathbb{B}$ of finite sets and bijection to the category $\mathbb{B}$. A linear species or $\mathbb{L}$-species (André Joyal [11], definition 12) is a functor from the $\mathbb{L}$ of finite linear orders and order-preserving bijection to the category $\mathbb{B}$. The exponential generating series of a species $\mathcal{F}$ ($\mathbb{B}$ or $\mathbb{L}$) is the series $\mathcal{F}(t) = \sum_{n\geq 0} f_n \frac{t^n}{n!}$, where $f_n = |\mathcal{F}[n]|$.

Operations sum, product, composition, and derivations that are compatible with the transition to exponential generating series are defined on the class of species ($\mathbb{B}$ or $\mathbb{L}$).
However, integration is only possible for $\mathbb{L}$-species. If $\mathcal{F}$ is an $\mathbb{L}$-species, then the integral of $\mathcal{F}$, denoted $\int \mathcal{F}$, is defined for every finite totally ordered set l by $(\int \mathcal{F})(l) = \emptyset$ if $l = \emptyset$, and $(\mathcal{F})(l) = \mathcal{F}(l \setminus \{\min l\})$ if $l \neq \emptyset$. A mixed species is a functor from the category $\mathbb{L} \times \mathbb{B}$ to the category of finite sets and bijection.

The method for solving systems of differential equations is essentially due to P. Leroux and G.X. Viennot in 14). In full generality, the data of an $\mathbb{L}$-species or any $\mathbb{B}$-species $\mathcal{F}$ allow us to write the combinatorial differential equation:

$$Y' = \mathcal{F}(Y); \quad Y(0) = Z \tag{3}$$

where Z is a sort of point representing the initial condition. This equation may be written in integral form:

$$Y(T, Z) = Z + \int_0^T \mathcal{F}(Y(X, Z))\, dX \tag{4}$$

This integral equation is interpreted by Figure 1, which is an iterative process for constructing the combinatorial solution of equation (3).

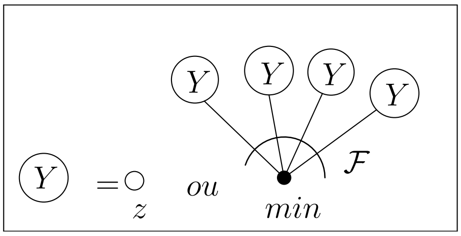

Figure 1: Integral equation

The general solution is the increasing $\mathcal{F}$-enriched tree $A_F(T, Z)$, who is a $\mathbb{L}$-species if $\mathcal{F}$ is a $\mathbb{L}$-species, and a mixed species if $\mathcal{F}$ is a $\mathbb{B}$-species. We have:

$$A_F(T, Z) = \exp(T\mathcal{D})(Z) = \sum_{n\geq 0} \frac{(T\mathcal{D})^n}{n!}(Z) \tag{5}$$

where $\mathcal{D} = \mathcal{F}(Z)\frac{d}{dZ}$ is the combinatorial differential operator associated with (3).

An analytical initial condition of the form $y(0) = x$ is combinatorially translated as $Y(0) = 1_x$, where 1_x is the species of the empty set, weighted by x. If $\mathcal{F}$ is a $\mathbb{B}$-species, the combinatorial differential equation $Y' = \mathcal{F}(Y), Y(0) = 1_x$ makes sense and its solution is the weighted $\mathbb{L}$-species $\mathbb{A}_{\mathcal{F}}(T) = T_{X=x}(A_{\mathcal{F}}(T,X))$ of the types with respect to the variable X of $A_{\mathcal{F}}(T,X)$ (see Randrianirina [15] theorem 6.3). Its generating series $y(t) = \mathbb{A}_{\mathcal{F}}(t)$ is generally the solution of the differential equation:

$$y'(t) = Z_{\mathcal{F}}(y(t); x, x^2, x^3, \cdots), \quad y(0) = x$$

where $Z_{\mathcal{F}}$ is the cycle index series of $\mathcal{F}$. If $\mathcal{F}$ is asymmetric, this equation becomes:

$$y'(t) = \mathcal{F}(y), \quad y(0) = x.$$

These results can be generalized to the system of combinatorial differential equations:

$$Y_i' = \mathcal{F}_i(Y_1, Y_2, \ldots, Y_k), \quad Y_i(0) = Z_i, \quad 1 \le i \le k \tag{6}$$

The solution of system (6) is the k-tuple of $\overrightarrow{\mathcal{F}}$-enriched increasing trees $\overrightarrow{\mathcal{A}}_{\overrightarrow{\mathcal{F}}} = (\mathcal{A}_{\overrightarrow{\mathcal{F}},1}, \cdots, \mathcal{A}_{\overrightarrow{\mathcal{F}},1})$, where $\mathcal{A}_{\overrightarrow{\mathcal{F}},i}$ is the solution of the equation $Y_i' = \mathcal{F}_i(Y_1, \cdots, Y_k), Y_i(0) = X_i$. Throughout the remainder of this paper, we assume that each $\mathcal{F}_i$ is an asymmetric $\mathbb{B}$-species.

According to Chen [9] (see also Dumont [8]) a context-free grammar G on X is a map from X to $\mathbb{C}[X]$, where $X = \{x_1, x_2, \cdots\}$ be an alphabet and $\mathbb{C}[X]$ a commutative algebra of polynomials in the letters x_i.

For each grammar G, we associate a differential operator

$$\mathcal{G} = \sum_{x \in X} G(x) \frac{\partial}{\partial x}.$$

satisfying $\mathcal{G}(f+g) = \mathcal{G}(f) + \mathcal{D}(g)$ and $\mathcal{G}(fg) = \mathcal{G}(f)g + f\mathcal{G}(g)$, for all $g \in \mathbb{C}[X]$.

For $u \in C[X]$, we associate an exponential generating function

$$\mathrm{Gen}(u,t) := \sum_{n \ge 0} \mathcal{G}^n(u) \frac{t^n}{n!}.$$

Thus, for all $u, v \in C[X]$,

$$\begin{gathered} \mathrm{Gen}(u+v,t) = \mathrm{Gen}(u,t) + \mathrm{Gen}(v,t), \quad \mathrm{Gen}(uv,t) = \mathrm{Gen}(u,t)\mathrm{Gen}(v,t) \\ \text{and} \quad \tfrac{\partial}{\partial t}\mathrm{Gen}(u,t) = \mathcal{G}\Big(\mathrm{Gen}(u,t)\Big). \end{gathered} \tag{7}$$

Let $X = \{x_1, x_2, \cdots, x_k\}$ be the alphabet. We have the following proposition, announced by Dumont in [8] and proved by Randrianirina in [15] (see also [16]):

Proposition 1.1. *Let $\vec{y}(t) = (y_i(t))_{i=1,\ldots,k}$ be the solution of the analytic differential system*

$$y_i'(t) = G_i(\vec{y}(t)), \qquad y_i(0) = x_i, \quad i = 1, \ldots, k. \tag{8}$$

Then for each i, $\mathrm{Gen}(x_i, t) = y_i(t)$.

The following theorem is proved by Randrianirina in [15]:

Theorem 1.1. *Given asymmetric species $(G_i)_{i=1,\ldots,k}$, the following data are equivalent:*

1. *the combinatorial differential system*

$$Y_i' = G_i(\vec{Y}); \qquad Y_i(0) = X_i, \quad i = 1, \ldots, k; \tag{9}$$

2. *the associated combinatorial differential operator*

$$\mathcal{G} = \sum_{i=1}^{k} G_i(\vec{X}) \frac{\partial}{\partial X_i}; \tag{10}$$

3. *William Chen grammar, where $G_i(\vec{x})$ is the generating series of G_i:*

$$G = \{x_i \mapsto G_i(\vec{x}); \qquad i = 1, \ldots, k\}; \tag{11}$$

4. *the analytic differential system*

$$y_i' = G_i(\vec{y}(t)); \qquad y_i(0) = x_i, \quad i = 1, \ldots, k. \tag{12}$$

The combinatorial differential operator $\mathcal{G}$ makes it possible to construct the combinatorial solutions of system (9). The solutions $(y_i(t))$ of the system of analytic differential equations (12) are the generating series of the isomorphism types of these solutions. And for all $i \in [k]$, $y_i(t) = \mathrm{Gen}(x_i, t)$.

2 GPK numbers and the grammar of Hao et al.

Let us consider the sequence $\Big(T(n,k)\Big)_{n,k}$ satisfying (1):

$$T(n,k) = (a_2 n + a_1 k + a_0) T(n-1,k) + (b_2 n + b_1 k + b_0) T(n-1,k-1).$$

Let G be the grammar

$$G = \{\, u \to u^{b_1+b_2+1} v^{a_1+a_2}, \ \ v \to u^{b_2} v^{a_2+1} \,\}, \tag{13}$$

so that the associated differential operator is

$$\mathcal{G} = u^{b_1+b_2+1} v^{a_1+a_2} \frac{\partial}{\partial u} + u^{b_2} v^{a_2+1} \frac{\partial}{\partial v}.$$

Hao et al. [1] proved the following result.

Proposition 2.1. *if $a_1 \geq 0$, $a_1 + a_2 \geq 0$, $a_1 + a_3 \geq 0$, $b_1 \geq 0$, $b_1 + b_2 \geq 0$ and $b_1 + b_2 + b_3 \geq 0$, then:*

$$\mathcal{G}^n(u^{b_0+b_1+b_2} v^{a_0+a_2}) = \sum_{k=0}^{n} T(n,k) u^{b_2 n + b_1 k + b_0 + b_1 + b_2} v^{a_2 n + a_1 k + a_0 + a_2}. \tag{14}$$

Example 2.1 (The r-Whitney-Eulerian numbers $A_{m,r}(n,k)$). In Foata [5], ${}^{r}A(n,k)$ is defined as the number of $\sigma \in S_n$ having k r-excedances, where $j \in [n]$ is an r-excedance of σ if $j+r \leq \sigma(j)$. Here we use the notation $A_r(n,k)$. The author proved that these numbers satisfy the recurrence relation:

$$A_r(n,k) = (k+r)A_r(n-1,k) + (n-k+1-r)A_r(n-1,k-1). \tag{15}$$

Riordan ([6]) (also in Maier [7]) discussed interpretations of the numbers satisfying relation (15), such as statistics of r-descents. This is a generalization of the Eulerian numbers $A_r(n,k) = \left\langle {n \atop k} \right\rangle$. It is also the number of $\sigma \in S_n$ having $n-r-k$ indices $i \in [n-1]$ satisfying $\sigma(i) < \sigma(i+1)$ and $\sigma(i) < n-r$. To generalize, first, we neglect the fact that this definition requires $r < n$. We start with r completely arbitrary. Second, we consider that each $i \in [n]$ may have m different types. These ideas inspire the definition of the r-Whitney-Eulerian numbers $A_{m,r}(n,k)$. A combinatorial interpretation is given in Thamrongpairoj [24].
These numbers are given by the recurrence relation ($A_{m,r}(n,k) = 0$ if $n < \max(0,k)$):

$$A_{m,r}(0,0) = 1 \quad \text{and} \quad A_{m,r}(n,k) = (mk+r)A_{m,r}(n-1,k) + \Big(mn-mk+m-r\Big)A_{m,r}(n-1,k-1).$$

The grammar of Hao associated is then

$$G = \{u \to uv^m, \\ v \to u^m v\};$$

and the system of differential equations is

$$\begin{cases} U' = & UV^m, \quad U(0) = u, \\ V' = & U^m V, \quad V(0) = v. \end{cases} \tag{16}$$

First, let's see the analytic solution

$$\begin{cases} U(t) = \left(\dfrac{(u^m - v^m)u^m}{u^m - v^m e^{(u^m - v^m)mt}} \right)^{\frac{1}{m}}, \\ V(t) = \left(\dfrac{(v^m - u^m)v^m}{v^m - u^m e^{(v^m - u^m)mt}} \right)^{\frac{1}{m}}. \end{cases} \tag{17}$$

Then, we have

$$Gen(u^{m-r}v^r, t) = \sum_{n \in \mathbb{N}} \mathcal{G}^n(u^{m-r}v^r)\frac{t^n}{n!} = \Big(U(t)\Big)^m \times \left(\frac{V(t)}{U(t)}\right)^r.$$

So the first thing that Theorem 1.1 and the Proposition 2.1 give us

$$\sum_{n \in \mathbb{N}} \left(\sum_{k=0}^{n} A_{m,r}(n,k) u^{mn-mk} v^{mk} \right) \frac{t^n}{n!} = \frac{(u^m - v^m)e^{(u^m - v^m)rt}}{u^m - v^m e^{(u^m - v^m)mt}}. \tag{18}$$

This equation can be used to get many things. For instance

$$\sum_{k=0}^{n} A_{m,r}(n,k) = m^n n!; \tag{19}$$

$$\sum_{k=0}^{n}(-1)^k A_{m,r}(n,k) = 2^n \sum_{k}\binom{n}{k} m^k E_k(0) r^{n-k}; \tag{20}$$

$$\sum_{k\in\mathbb{N}}\left(\sum_{n\in\mathbb{N}} A_{m,r}(n,k)\frac{t^n}{n!}\right)x^k = \frac{(1-x)e^{r(1-x)t}}{1-xe^{(1-x)mt}}. \tag{21}$$

We can get also the explicit formula of $A_{m,r}(n,k)$. Indeed,

$$\begin{aligned}
\frac{(1-x)e^{r(1-x)t}}{1-xe^{(1-x)mt}} =&(1-x)\sum_{j\geq 0} x^j e^{(1-x)(mj+r)t}\\
=&(1-x)\sum_{j\geq 0} x^j \sum_{n\geq 0}\frac{\big((1-x)(mj+r)t\big)^n}{n!}\\
=&\sum_{n\geq 0}\left((1-x)^{n+1}\sum_{j\geq 0} x^j (mj+r)^n\right)\frac{t^n}{n!}\\
=&\sum_{n\geq 0}\left(\sum_{l\geq 0}\binom{n+1}{l}x^l\sum_{j\geq 0}(mj+r)^n x^j\right)\frac{t^n}{n!}\\
=&\sum_{k\in\mathbb{N}}\sum_{n\in\mathbb{N}}\left(\sum_{j=0}^{j=k}(-1)^j\binom{n+1}{j}\big(m(k-j)+r\big)^n\right)\frac{t^n}{n!}x^k. \qquad (22)
\end{aligned}$$

(21) and (22) imply

$$A_{m,r}(n,k) = \sum_{j=0}^{j=k}(-1)^j\binom{n+1}{j}\big(m(k-j)+r\big)^n. \tag{23}$$

For $m \geq r$, the grammar of Hao G can give us ideas about a structure that can interpret the numbers $\big(A_{m,r}(n,k)\big)_{n,k\geq 0}$. For example, let $m=3$ and $r=2$. The grammar is then

$$\begin{aligned} G = \{u &\to uv^3\\ v &\to u^3v\}, \end{aligned}$$

and the associated differential operator is

$$\mathcal{G} = uv^3\frac{\partial}{\partial u} + u^3v\frac{\partial}{\partial v}.$$

$$\mathcal{G}^0 uv^2 = uv^2$$

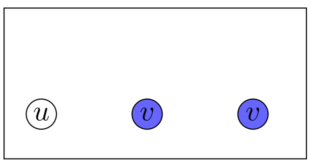

Figure 2: $\mathcal{G}^n(u^{m-r}v^r)$-structure for $n=0$

$$\mathcal{G}uv^2 = uv^5 + 2u^4v^2$$

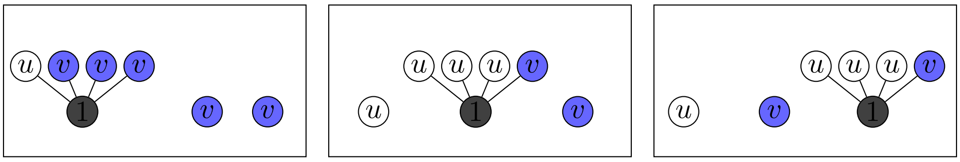

Figure 3: $\mathcal{G}^n(u^{m-r}v^r)$-structures for $n = 1$

$$\mathcal{G}^2uv^2 = uv^8 + 13u^4v^5 + 4u^7v^2$$

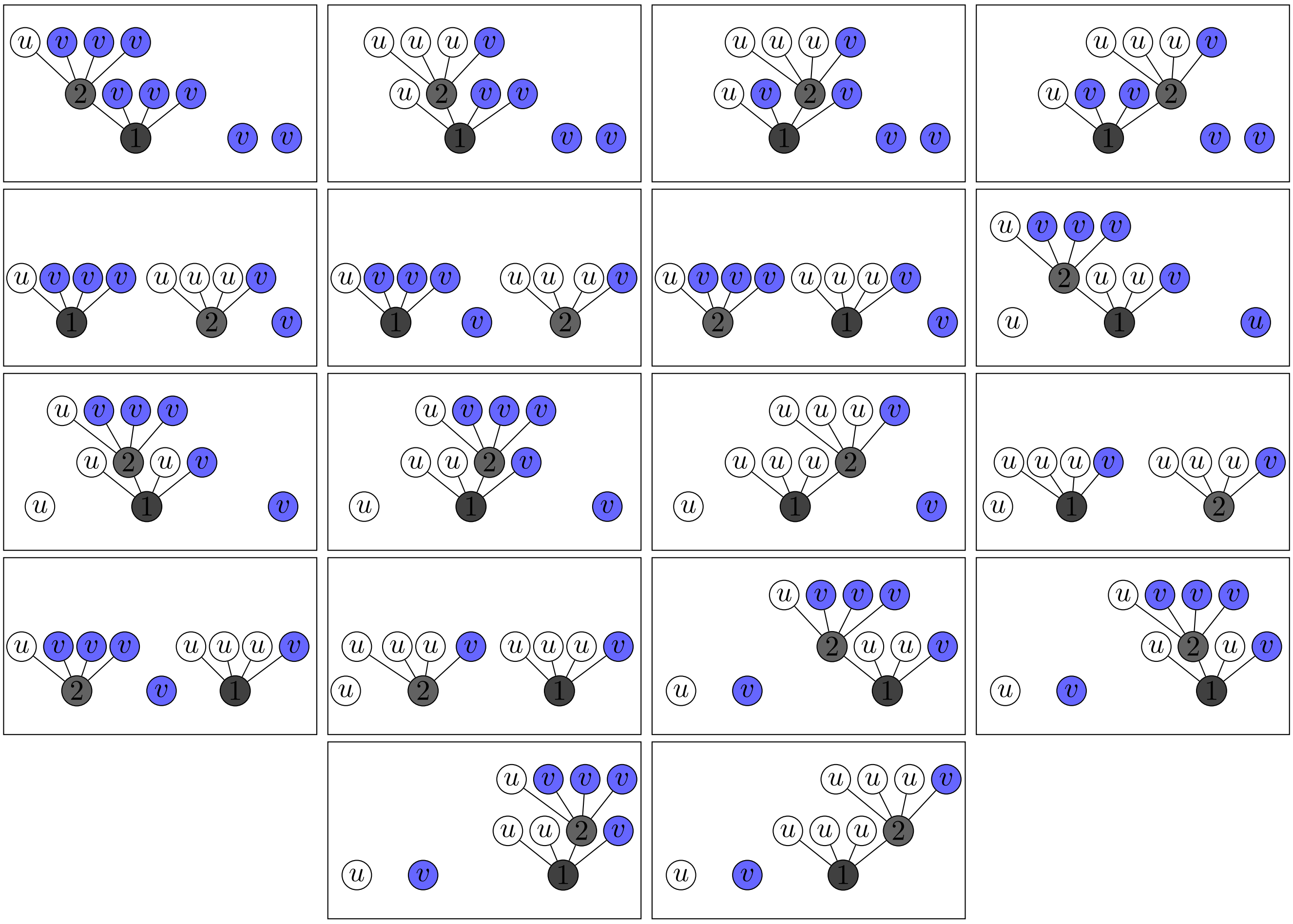

Figure 4: $\mathcal{G}^n(u^{m-r}v^r)$-structures for $n = 2$

From Proposition 2.1, $A_{m,r}(n, k)$ is the number of $\mathcal{G}^n(u^{m-r}v^r)$-structures having $km+r$ v-leaves (blue in our Figures 2, 3, 4). Moreover, we notice that a $\mathcal{G}^n(u^{m-r}v^r)$-structure has $f_n = m(n+1)$ leaves, and if a_n denotes the total number of $\mathcal{G}^n(u^{m-r}v^r)$-structures, then $a_0 = 1$ and for $n \geq 1$, $a_n = a_{n-1} \times f_{n-1} = nma_{n-1}$. Therefore,

$$a_n = n!\, m^n,$$

which reaffirms relation (19).

Example 2.2. *Consider the sequence* $\Big(F(n,k)\Big)_{n,k\geq 0}$ *satisfying* (1)*, with* $b_{n,k}=1$ *and* $a_{n,k}=a_0+a_1k$*,* $a_1\neq 0$*. The system of differential equations associated with the grammar* G *in* (13) *is therefore*

$$\begin{cases} U' &= UV^{a_1}, \quad U(0)=u,\\ V' &= V, \qquad\quad V(0)=v. \end{cases}$$

The analytic solution is

$$\begin{cases} U(t) &= u\exp\left(\frac{v^{a_1}}{a_1}\Big(e^{a_1t}-1\Big)\right),\\ V(t) &= ve^{t}. \end{cases} \tag{24}$$

Recall that the Bell polynomial is defined by

$$B_n(\lambda)=\sum_{k=0}^{n}S(n,k)\lambda^k.$$

The classical fact concerning the Stirling numbers then gives

$$\sum_{n\geq 0}B_n(\lambda)\frac{t^n}{n!}=\exp\Big(\lambda(e^t-1)\Big). \tag{25}$$

Equations (24) and (25) imply

$$U(t)\Big(V(t)\Big)^{a_0}=uv^{a_0}\sum_{n\geq 0}\left(\sum_{k=0}^{n}\binom{n}{k}a_1^ka_0^{n-k}B_k\Big(\frac{v^{a_1}}{a_1}\Big)\right)\frac{t^n}{n!}. \tag{26}$$

On the other hand, Proposition 2.1 states that

$$\mathcal{G}^n(uv^{a_0})=\sum_{k=0}^{n}F(n,k)\,uv^{a_1k+a_0}. \tag{27}$$

Now, combining these facts with equation (7) and Proposition 1.1, we obtain

$$\sum_{k\geq 0}F(n,k)\alpha^k=\sum_{k\geq 0}\binom{n}{k}a_1^ka_0^{n-k}B_k\Big(\frac{\alpha}{a_1}\Big), \tag{28}$$

by substituting $\alpha=v^{a_1}$.
Returning to the expression for $B_k(x)$, we have

$$F(n,k)=\sum_{j=k}^{n}\binom{n}{j}a_0^{n-j}a_1^{j-k}S(j,k). \tag{29}$$

Next, consider $\mathcal{G}^n(uv^{a_0})$, with

$$\mathcal{G}=uv^{a_1}\frac{\partial}{\partial u}+v\frac{\partial}{\partial v}.$$

We now look at the example where $a_0=2$ and $a_1=2$.

$$\mathcal{G}^0 uv^2 = uv^2$$

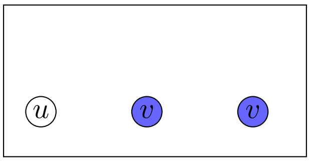

Figure 5: $\mathcal{G}^n(uv^{a_0})$-structure for $n = 0$

$$\mathcal{G}uv^2 = uv^4 + 2uv^2$$

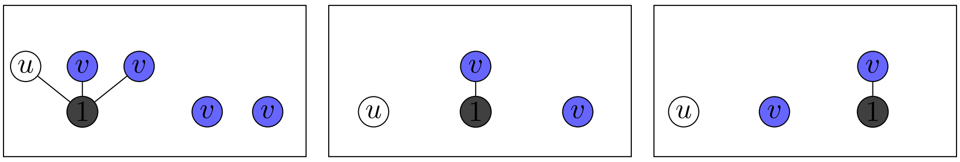

Figure 6: $\mathcal{G}^n(uv^{a_0})$-structures for $n = 1$

$$\mathcal{G}^2 uv^2 = uv^6 + 4uv^4 + 2uv^4 + 4uv^2$$

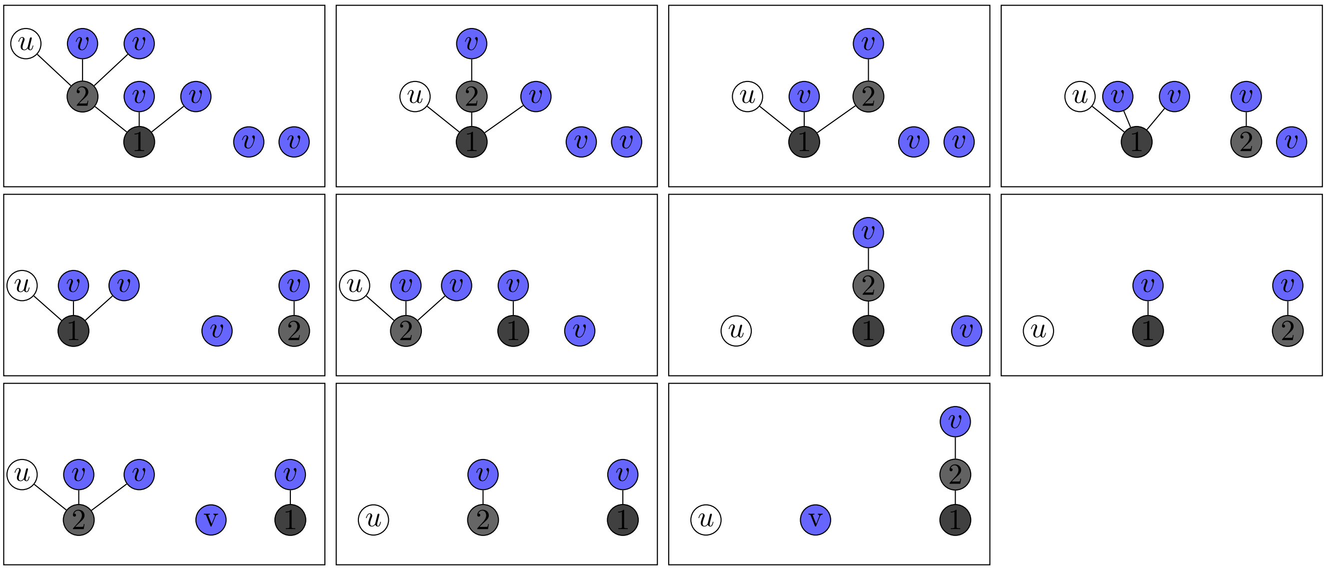

Figure 7: $\mathcal{G}^n(uv^{a_0})$-structures for $n = 2$

From equation (27), $F(n,k)$ counts the number of $\mathcal{G}^n(uv^{a_0})$-structures having $a_1 k + a_0$ v-leaves (blue in our Figures 5, 6, 7).

Example 2.3. *Consider the sequence* $\big(F(n,k)\big)_{n,k\geq 0}$ *satisfying* (1)*, with* $b_{n,k} = 1$ *and* $a_{n,k} = a_0 + a_2 n$*,* $a_2 \neq 0$*. The system of differential equations associated with the grammar* G_2 *in* (13) *is therefore*

$$\begin{cases} U' &= UV^{a_2}, \quad U(0) = u, \\ V' &= V^{a_2+1}, \quad V(0) = v. \end{cases}$$

The analytic solution is

$$\begin{cases} U(t) &= \frac{u}{v}\left(v^{-a_2} - a_2 t\right)^{-\frac{1}{a_2}}, \\ V(t) &= \left(v^{-a_2} - a_2 t\right)^{-\frac{1}{a_2}}. \end{cases} \tag{30}$$

Thus, we have

$$U(t)\Big(V(t)\Big)^{a_0+a_2} = \frac{u}{v}\left(v^{-a_2} - a_2 t\right)^{-\alpha} = uv^{a_0+a_2}\sum_{n\geq 0}(\alpha)^{(\underline{n})} a_2^n v^{a_2 n}\frac{t^n}{n!},$$

where $\alpha = \frac{1+a_0+a_2}{a_2}$.
Again, as before, using Proposition 2.1, we deduce

$$\sum_{n\geq 0}\sum_{k=0}^{n} F(n,k)\, uv^{a_0+a_2+a_2 n}\frac{t^n}{n!} = uv^{a_0+a_2}\sum_{n\geq 0}(\alpha)^{(\underline{n})} a_2^n v^{a_2 n}\frac{t^n}{n!}.$$

This equation implies that

$$\sum_{k=0}^{n} F(n,k) = (\alpha)^{(\underline{n})} a_2^n = \left(\frac{1+a_0+a_2}{a_2}\right)^{(\underline{n})} a_2^n.$$

Remark 2.1. Briefly, Proposition 2.1 states that

$$\mathcal{G}^n(uv^{a_0+a_2}) = \sum_k T(n,k)\, uv^{a_2 n + a_2 + a_0},$$

which does not allow us to distinguish the individual $T(n,k)$ according to the number of v-leaves; all $\mathcal{G}^n(uv^{a_0+a_2})$-structures have the same number of leaves: one u-leaf and $a_2 n + a_2 + a_0$ v-leaves.

Therefore, it is necessary to introduce additional properties to determine the individual $T(n,k)$. We now briefly discuss a certain type of grammar: the type (E) grammar, introduced in Randrianirina [17].

Definition 2.1 ([17])**.** *A type (E) grammar is a grammar G on $\{z\}\cup X$, with $z \notin X$, defined by*

$$\begin{aligned} G = \{z &\to zg(\vec{x}) \\ &x_i \to h_i(\vec{x}) : \quad x_i \in X\}, \end{aligned}$$

where $g(\vec{x})$ is a polynomial in $C[X]$.

We associate to this grammar a system of differential equations. By solving it combinatorially, the $\mathcal{Z}$-structure is a species of structures as illustrated in Figure 8. Hence, $\mathcal{Z}$ is the set of $\mathcal{G}$-structures, with

$$\mathcal{G} = \int g\big(\vec{\mathcal{X}}\big).$$

Now consider the blue line, which we call the backbone. For the language in Randrianirina [17], each point on the backbone is associated with a connected component, and the author associates the species $\mathcal{Z}$ to a sequence $\Big(Z(n,k)\Big)_{n,k\geq 0}$, where $Z(n,k)$ is the number of $\mathcal{Z}$-structures on n

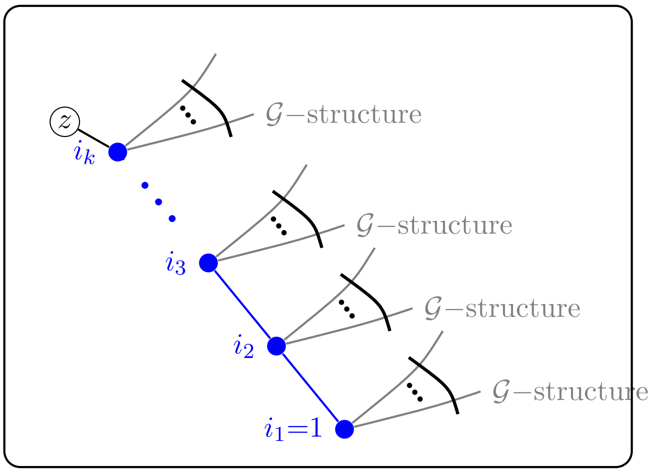

Figure 8: $\mathcal{Z}$-structure in terms of $\mathcal{G}$-structure.

points having k connected components.
Now consider a sequence $\big(F(n,k)\big)_{n,k\geq 0}$ satisfying (1) with $a_{n,k} = a_2 n + a_1 k + a_0$, $b_{n,k} = 1$, and the usual initial conditions. The grammar G from (13) associated with this sequence is thus

$$G = \{u \to uv^{a_1+a_2}; \\ v \to v^{a_2+1}.\}$$

This grammar is of type (E), and we have the following important remark.

Proposition 2.2. *Every $\mathcal{G}^n(uv^{a_0+a_2})$-structure s is canonically decomposed as $s = (s_1, s_2)$, where s_1 is a $\mathcal{U}$-structure and s_2 is a $\mathcal{V}^{a_0+a_2}$-structure. Therefore, $F(n,k)$ is the number of $\mathcal{G}_2^n(uv^{a_0+a_2})$-structures s such that s_1 has k connected components.*

Proof. First, Proposition 2.1 ensures that $F(n,k)$ is the number of $\mathcal{G}^n(uv^{a_0+a_2})$-structures having $a_2n{+}a_1k{+}a_0{+}a_2$ v-leaves. Now, let $G(n,k)$ denote the number of $\mathcal{G}^n(uv^{a_0+a_2})$-structures s whose s_1 has k connected components.
We proceed by induction. For $n = 0$, the unique structure has $a_0 + a_2$ v-leaves, so the relation holds trivially. For $n \geq 1$, assume the relation holds up to $n-1$, i.e., $T(n,k) = G(n,k)$, and show it for n. Consider a $\mathcal{G}_2^n(uv^{a_0+a_2})$-structure s with k connected components. We focus on the position of point n, with two possible cases:

- Case 1: point n is on the spine (at i_k). This occurs if and only if the last derivation was applied to the unique u-leaf. This last step increases the number of connected components.

- Case 2: point n is not on the spine. Then the last derivation was applied to one of the v-leaves. This last step does not increase the number of connected components. Using the induction hypothesis and Proposition 2.1, there are $a_2(n-1) + a_1k + a_0$ choices for this last step.

This correspondence ensures that $\big(G(n,k)\big)_{n,k\in\mathbb{N}}$ satisfies

$$G(n,k) = G(n-1,k-1) + (a_2n + a_1k + a_0)G(n-1,k),$$

which completes the proof. $\square$

Corollary 2.1. *Let $(T(n,k))$ be a sequence satisfying the usual initial conditions with*

$$T(n,k) = T(n-1,k-1) + (a_2 n + a_1 k - a_2)T(n-1,k),$$

and consider the combinatorial differential equation system

$$\begin{cases} U' = & UV^{a_1+a_2}, \quad U(0) = u, \\ V' = & V^{a_2+1}, \quad V(0) = v. \end{cases}$$

Then $T(n,k)$ represents the number of $\mathcal{U}$-structures with k connected components.

Theorem 2.1. *Let $\big(F(n,k)\big)$ be a sequence satisfying* (1) *with $a_{n,k} = a_2 n + a_1 k + a_0$, $b_{n,k} = 1$, and usual initial conditions. If $a_1 \neq 0$ and $a_2 \neq 0$, then*

$$F(n,k) = \frac{1}{a_1^k\, k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \prod_{r=1}^{n} (a_0 + a_1 j + r a_2). \tag{31}$$

Proof. The grammar G, given in (13), is

$$G = \{u \to uv^{a_1+a_2}, \\ v \to v^{a_2+1}\}.$$

Consider the system

$$\begin{cases} U' = UV^{a_1+a_2}, \quad U(0) = u, \\ V' = V^{a_2+1}, \quad V(0) = v. \end{cases}$$

By separation of variables, we get

$$V(t) = \left(v^{-a_2} - a_2 t\right)^{-1/a_2}, \quad U(t) = u \exp\left(\frac{(v^{-a_2} - a_2 t)^{-a_1/a_2} - v^{a_1}}{a_1}\right),$$

valid as long as $v^{-a_2} - a_2 t \neq 0$. Then

$$U(t)V(t)^{a_0+a_2} = uv^{a_0+a_2}(1 - a_2 t\, v^{a_2})^{-(a_0+a_2)/a_2} \exp\left(\frac{v^{a_1}}{a_1}\left((1 - a_2 t\, v^{a_2})^{-a_1/a_2} - 1\right)\right).$$

Expanding the exponential and the binomial, we obtain

$$U(t)V(t)^{a_0+a_2} = \sum_{k \geq 0} \frac{1}{k!} \left(\frac{v^{a_1}}{a_1}\right)^k \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} (1 - a_2 t\, v^{a_2})^{-(a_0+a_2+a_1 j)/a_2}.$$

Moreover, for $n \geq 0$,

$$(1 - a_2 t\, v^{a_2})^{-\beta} = \sum_{n \geq 0} \left(\prod_{r=1}^{n} (\beta + r - 1)\right) \frac{(a_2 t\, v^{a_2})^n}{n!}.$$

Therefore,

$$U(t)V(t)^{a_0+a_2} = \sum_{k \geq 0} \frac{1}{a_1^k k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \prod_{r=1}^{n} (a_0 + a_1 j + r a_2) \frac{t^n}{n!}. \tag{32}$$

On the other hand, Proposition 1.1 ensures

$$U(t)V(t)^{a_0+a_2} = Gen(uv^{a_0+a_2}, t) = \sum_{n\geq 0}\left(\sum_{k=0}^{n} T(n,k)\, u\, v^{a_2 n+a_1 k+a_2+a_0}\right)\frac{t^n}{n!}. \tag{33}$$

Comparing (32) and (33) yields

$$F(n,k) = \frac{1}{a_1^k k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\prod_{r=1}^{n}(a_0+a_1 j+ra_2).$$

□

3 Applications

We have already seen an example involving the r-Whitney–Eulerian numbers. We now present several further applications. Some of these yield improved formulations of known results, while others provide new contributions.

3.1 Descents in Stirling r-permutations

Xiao He ([25]) studied $B^{(r)}(n,k)$, the numbers of r-permutations on $[n]$ with k descents. He proved that for all k and $n > 0$,

$$B^{(r)}(n,k) = \Big(rn-k+(1-r)\Big)B^{(r)}(n-1,k-1)+(k+1)B^{(r)}(n-1,k), \tag{34}$$

with $B^{(r)}(n,0) = 1$ pour n$\geq$ 1 et $B^{(r)}(n,k) = 0$ if $n \leq k$ ou $k < 0$.
First, the grammar of Hao associated is then

$$\begin{aligned} G = \{x &\to x^r y,\\ y &\to x^r y\}. \end{aligned}$$

Definition 3.1 (Full r-ary Tree)**.** *A* full r-ary tree *is a rooted tree in which every internal node has exactly r children, where $r \geq 1$ is fixed. Nodes with no children are called* leaves. *Thus, in a full r-ary tree, each internal node has exactly r children and each leaf has* 0 *children.*

We denote $\mathcal{A}^{(r)}$ the structure associated to full r-ary tree. It can be defined also recursively. For $n=0$: $\mathcal{A}^{(r)} = \emptyset$ and for $n \geq 0$: $\mathcal{A}^{(r)} = \left(p, \mathcal{A}_1^{(r)}, \mathcal{A}_2^{(r)} \cdots, \mathcal{A}_r^{(r)}\right)$, with p is a point called root and the $\left(\mathcal{A}_i^{(r)}\right)$ are full r-ary trees on k_i points with $\sum_{i=1}^{i=r} k_i = n-1$.
Here, we consider also an order on each children and think that the last among the r children is called cadet.

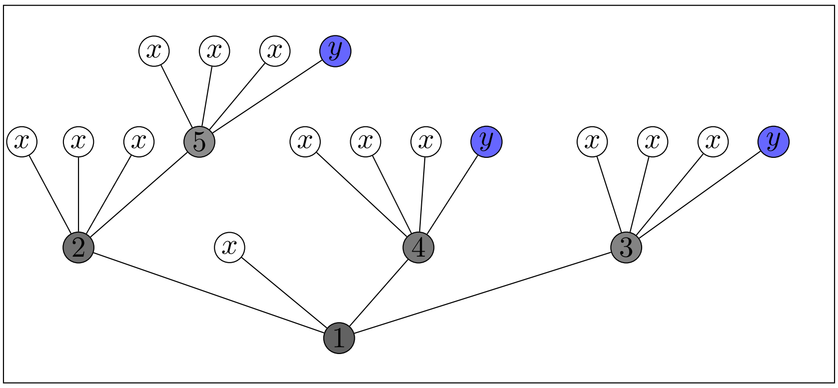

Figure 9: A full 4-ary true for $n = 5$

In Figure 9, the tree has 16 leaves and the blue nodes (y) are cadets; there are 3 cadets.

Proposition 3.1. *$B^{(r)}(n, k)$ counts the number of $\mathcal{A}^{(r+1)}$-structures on $[n]$ with $k + 1$ cadet leaves.*

Proof. First, the proposition 2.1 ensures us

$$\mathcal{G}^n(y) = \sum_{k=0}^{n} B^{(r)}(n, k)x^{nr-k}y^{k+1}. \tag{35}$$

Then $B^{(r)}(n, k)$ is the numbers of $\mathcal{G}^n(y)$-structure having $k + 1$ y-leaves. We notice also that $\mathcal{G}^n(y)$-structure is the same than $\mathcal{A}^{(r+1)}$-structure where the leaves are labeled x and y (only the last children is y). Thus we conclude. $\square$

Moreover, the system of differential equations associated to the grammar is

$$\begin{cases} X'(t) = X^r(t)Y(t), & X(0) = x \\ Y'(t) = X^r(t)Y(t), & Y(0) = y. \end{cases} \tag{36}$$

We apply know Proposition 1.1 (or Theorem 1.1) to the equation (35) to get the following.

Theorem 3.1. *If $\big(X(t), Y(t)\big)$ is the solution of the system of differential equations (36), then*

$$\sum_{n\geq 0}\Big(\sum_{k=0}^{n} B^{(r)}(n, k)x^{nr-k}y^{k+1}\Big)\frac{t^n}{n!} = Y(t). \tag{37}$$

3.1.1 Case where $r = 2$

We write B for $B^{(2)}$, so (34)'s version is

$$B(n, k) = (k + 1)B(n - 1, k) + (2n - k - 1)B(n - 1, k - 1). \tag{38}$$

Define the formal series

$$T(z) := \sum_{n\geq 1} n^{n-1}\frac{z^n}{n!}.$$

One can compute, or use directly the fact about nonplane labelled trees in Flajolet([26]) to get

$$T(z) = ze^{T(z)} \implies T'(z) = \frac{e^{T(z)}}{1 - zT(z)} = \frac{e^{T(z)}}{1 - T(z)}. \tag{39}$$

Then, for $x \neq y$

$$\begin{cases} X(t) = \dfrac{x-y}{1 - T\Big(\frac{y}{x}\exp\Big(-\frac{y}{x} + (x-y)^2 t\Big)\Big)}, \\ Y(t) = \dfrac{(x-y)\, T\Big(\frac{y}{x}\exp\Big(-\frac{y}{x} + (x-y)^2 t\Big)\Big)}{1 - T\Big(\frac{y}{x}\exp\Big(-\frac{y}{x} + (x-y)^2 t\Big)\Big)} \end{cases} \tag{40}$$

solves the equation (36) if $r = 2$. We apply now the proposition 1.1, with equation (35) to have

Theorem 3.2. *For $x \neq y$*

$$\sum_{n\geq 0}\sum_{k=0}^{k=n} B(n,k) x^{2n-k} y^{k+1} \frac{t^n}{n!} = \frac{(x-y)\, T\Big(\frac{y}{x}\exp\Big(-\frac{y}{x} + (x-y)^2 t\Big)\Big)}{1 - T\Big(\frac{y}{x}\exp\Big(-\frac{y}{x} + (x-y)^2 t\Big)\Big)}. \tag{41}$$

Corollary 3.1. *For $y \neq 1$*

$$\sum_{n\geq 0}\sum_{k=0}^{k=n} B(n,k) y^{k+1} \frac{t^n}{n!} = \frac{(1-y)\, T(y\exp(-y + (1-y)^2 t))}{1 - T(y\exp(-y + (1-y)^2 t))}. \tag{42}$$

3.1.2 General case $r \geq 1$

Let's define first a combinatorial structure.

Definition 3.2. *Fix integers a, b, r with $b - a \in \mathbb{N}$.*

- *A tree may consist of a single leaf. Such a leaf can be colored in a different colors.*
- *Otherwise, a leaf can be expanded into an internal node. Each internal node produces either:*
  - *exactly r children, or*
  - *exactly $r+1$ children.*
- *Coloring rules:*
  - *Leaves are colored with a colors;*
  - *Internal nodes with r children can be colored in $(b-a)$ different ways;*
  - *Internal nodes with $r+1$ children carry no additional color.*
- *we label each internal node in such way it is croissant (increasing) meanly if node x is descend of node y then labbel of $x >$ label of y.*

Let c_n be the numbers of such tree with n internal nodes and $C_b^a(t) := \sum_{n\geq 1} c_n \frac{t^n}{n!}$ be the generating function. One can see is

$$\begin{cases} c_0 = a \\ c_{k+1} = \sum_{k_1+\cdots+k_{r+1}=k} \binom{k}{k_1,\ldots,k_{r+1}} c_{k_1}\cdots c_{k_{r+1}} + (b-a)\sum_{k_1+\cdots+k_r=k}\binom{k}{k_1,\ldots,k_r} c_{k_1}\cdots c_{k_r}. \end{cases} \tag{43}$$

Definition 3.3. *We define the formal series*

$$C_y(t) := \sum_{n\geq 1} f_n(y)\frac{t^n}{n!},$$

where

$$\begin{cases} f_0 = 1 \\ f_{k+1}(y) = \sum_{k_1+\cdots+k_{r+1}=k} \binom{k}{k_1,\ldots,k_{r+1}} f_{k_1}(y)\cdots f_{k_{r+1}}(y) + (y-1)\sum_{k_1+\cdots+k_r=k}\binom{k}{k_1,\ldots,k_r} f_{k_1}(y)\cdots f_{k_r}(y). \end{cases} \tag{44}$$

Analytically, we can check that

$$C_y'(t) = C_y(t)^{r+1} + (y-1)C_y(t)^r; \quad C_y(0) = 1. \tag{45}$$

The construction is just $C_y = C_y^1$.
The combinatorial integration with the definition 3.2 also allow us to say

$$(\mathcal{C}_b^a)' = (\mathcal{C}_b^a)^{r+1} + (b-a)(\mathcal{C}_b^a)^r \implies (C_b^a)'(t) = (C_b^a)^{r+1}(t) + (b-a)(C_b^a)^r(t). \tag{46}$$

So, $\big(C_y^x(t), C_y^x(t) + (y-x)\big)$ solves the system of differential equations (36). Thus, Theorem 3.1 gives us the following.

Theorem 3.3.

$$\sum_{n\geq 0}\Big(\sum_{k=0}^{n} B^{(r)}(n,k)x^{nr-k}y^{k+1}\Big)\frac{t^n}{n!} = C_y^x(t) + (y-x). \tag{47}$$

In particular

Theorem 3.4.

$$\sum_{n\geq 0}\Big(\sum_{k=0}^{n} B^{(r)}(n,k)y^{k+1}\Big)\frac{t^n}{n!} = C_y(t) + (y-1). \tag{48}$$

3.2 Case $b_2 = 0$

Many years ago, Wilf [21] (and also Neuwirth [19]) found the generating function for the case $b_2 = 0$, which yields an explicit formula. We will show here that these results can be seen quickly using grammar.

The Hao grammar associated is $u \to u^{b_1+1}v^{a_1+a_2}$, $v \to v^{a_2+1}$ with differential system

$$\begin{cases} U' = U^{b_1+1}V^{a_1+a_2}, & U(0) = u, \\ V' = V^{a_2+1}, & V(0) = v. \end{cases}$$

	$b_1 = 0$	$b_1 = -1$	$b_1 \neq 0, -1$
$a_2 = 0$	$V(t) = ve^t$ $U(t) = \begin{cases} u \exp\left(\frac{v^{a_1}}{a_1}(e^{a_1 t} - 1)\right), & a_1 \neq 0 \\ ue^t, & a_1 = 0 \end{cases}$	$V(t) = ve^t$ $U(t) = \begin{cases} u + \frac{v^{a_1}}{a_1}(e^{a_1 t} - 1), & a_1 \neq 0 \\ u + t, & a_1 = 0 \end{cases}$	$V(t) = ve^t$ $U(t) = \begin{cases} \left[u^{-b_1} - \frac{b_1 v^{a_1}}{a_1}(e^{a_1 t} - 1)\right]^{-1/b_1}, & a_1 \neq 0 \\ \left[u^{-b_1} - b_1 t\right]^{-1/b_1}, & a_1 = 0 \end{cases}$
$a_2 = -1$	$V(t) = v + t$ $U(t) = \begin{cases} u \exp\left(\frac{(v+t)^{a_1} - v^{a_1}}{a_1}\right), & a_1 \neq 0 \\ u\frac{v+t}{v}, & a_1 = 0 \end{cases}$	$V(t) = v + t$ $U(t) = \begin{cases} u + \frac{(v+t)^{a_1} - v^{a_1}}{a_1}, & a_1 \neq 0 \\ u + \ln\left(\frac{v+t}{v}\right), & a_1 = 0 \end{cases}$	$V(t) = v + t$ $U(t) = \begin{cases} \left[u^{-b_1} - \frac{b_1}{a_1}\left((v+t)^{a_1} - v^{a_1}\right)\right]^{-1/b_1}, & a_1 \neq 0 \\ \left[u^{-b_1} - b_1 \ln\left(\frac{v+t}{v}\right)\right]^{-1/b_1}, & a_1 = 0 \end{cases}$
$a_2 \neq 0, -1$	$V(t) = (v^{-a_2} - a_2 t)^{-1/a_2}$ $U(t) = \begin{cases} u \exp\left(\frac{(v^{-a_2} - a_2 t)^{-a_1/a_2} - v^{a_1}}{-a_1}\right), & a_1 \neq 0 \\ u(v^{-a_2} - a_2 t)^{-1/a_2} v, & a_1 = 0 \end{cases}$	$V(t) = (v^{-a_2} - a_2 t)^{-1/a_2}$ $U(t) = \begin{cases} u + \frac{(v^{-a_2} - a_2 t)^{-a_1/a_2} - v^{a_1}}{-a_1}, & a_1 \neq 0 \\ u - \frac{1}{a_2} \ln\left(\frac{v^{-a_2} - a_2 t}{v^{-a_2}}\right), & a_1 = 0 \end{cases}$	$V(t) = (v^{-a_2} - a_2 t)^{-1/a_2}$ $U(t) = \begin{cases} \left[u^{-b_1} - \frac{b_1}{a_1}\left((v^{-a_2} - a_2 t)^{-a_1/a_2} - v^{a_1}\right)\right]^{-1/b_1}, & a_1 \neq 0 \\ \left[u^{-b_1} + \frac{b_1}{a_2} \ln\left(\frac{v^{-a_2} - a_2 t}{v^{-a_2}}\right)\right]^{-1/b_1}, & a_1 = 0 \end{cases}$

This can be combine with Propositions 2.1 and 1.1 one

$$U(t)^{b_1+b_0} V(t)^{a_2+a_0} = \sum_{n \geq 0} \left(\sum_{k=0}^{n} T(n,k) u^{b_1 k + b_0 + b_1} v^{a_2 n + a_1 k + a_0 + a_2} \right) \frac{t^n}{n!}.$$

For instance, if we take the case where $a_2, b_1 \notin \{0, -1\}$, $a_1 \neq 0$ for $v = 1$:

$$\left(u^{-b_1} + \frac{b_1}{a_1} \left(1 - (1 - a_2 t)^{-a_1/a_2}\right) \right)^{-\frac{b_1+b_0}{b_1}} \cdot (1 - a_2 t)^{-\frac{a_2+a_0}{a_2}} = \sum_{n \geq 0} \left(\sum_{k=0}^{n} T(n,k) u^{b_1 k + b_0 + b_1} \right) \frac{t^n}{n!}.$$

We can change now $x = u^{b_1}$ and get the following.

Proposition 3.2. *If* $a_2, b_1 \notin \{0, -1\}$, $a_1 \neq 0$, *then*

$$\left(1 + x \frac{b_1}{a_1} \left(1 - (1 - a_2 t)^{-a_1/a_2}\right) \right)^{-\frac{b_1+b_0}{b_1}} (1 - a_2 t)^{-\frac{a_2+a_0}{a_2}} = \sum_{n \geq 0} \left(\sum_{k=0}^{n} T(n,k)\, x^k \right) \frac{t^n}{n!}.$$

This is similar to what Herbert [21] has.

3.3 Case $a_1 + a_2 = 0$

The Hao grammar associated is $u \to u^{b_1+b_2+1}$, $v \to u^{b_2} v^{a_2+1}$ with differential system

$$\begin{cases} U' = U^{b_1+b_2+1}, & U(0) = u, \\ V' = U^{b_2} V^{a_2+1}, & V(0) = v. \end{cases}$$

	$a_2 = 0$	$a_2 = -1$	$a_2 \neq 0, -1$
$b_1 + b_2 = 0$	$U(t) = ue^t$ $V(t) = \begin{cases} v \exp\left(\frac{u^{b_2}}{b_2}(e^{b_2 t} - 1)\right), \\ \qquad b_2 \neq 0 \\ ve^t, \quad b_2 = 0 \end{cases}$	$U(t) = ue^t$ $V(t) = \begin{cases} v + \frac{u^{b_2}}{b_2}(e^{b_2 t} - 1), \\ \qquad b_2 \neq 0 \\ v + t, \quad b_2 = 0 \end{cases}$	$U(t) = ue^t$ $V(t) = \begin{cases} \left[v^{-a_2} - \frac{a_2 u^{b_2}}{b_2}(e^{b_2 t} - 1)\right]^{-1/a_2}, \\ \qquad b_2 \neq 0 \\ \left[v^{-a_2} - a_2 t\right]^{-1/a_2}, \quad b_2 = 0 \end{cases}$
$b_1 + b_2 = -1$	$U(t) = u + t$ $V(t) = \begin{cases} v \exp\left(\frac{(u+t)^{b_2+1} - u^{b_2+1}}{b_2+1}\right), \\ \qquad b_2 \neq -1 \\ v\frac{u+t}{u}, \quad b_2 = -1 \end{cases}$	$U(t) = u + t$ $V(t) = \begin{cases} v + \frac{(u+t)^{b_2+1} - u^{b_2+1}}{b_2+1}, \\ \qquad b_2 \neq -1 \\ v + \ln\left(\frac{u+t}{u}\right), \quad b_2 = -1 \end{cases}$	$U(t) = u + t$ $V(t) = \begin{cases} \left[v^{-a_2} - \frac{a_2}{b_2+1}\left((u+t)^{b_2+1} - u^{b_2+1}\right)\right]^{-1/a_2}, \\ \qquad b_2 \neq -1 \\ \left[v^{-a_2} - a_2 \ln\left(\frac{u+t}{u}\right)\right]^{-1/a_2}, \quad b_2 = -1 \end{cases}$
$b_1 + b_2 \neq 0, -1$	$U(t) = (u^{-(b_1+b_2)} - (b_1+b_2)t)^{-1/(b_1+b_2)}$ $V(t) = \begin{cases} v \exp\left(\frac{(u^{-(b_1+b_2)} - (b_1+b_2)t)^{1-\frac{b_2}{b_1+b_2}} - u^{b_2}}{b_1}\right), \\ \qquad b_1 \neq 0 \\ v \exp\left(\ln \frac{u^{-(b_1+b_2)} - (b_1+b_2)t}{u^{-(b_1+b_2)}}\right), \quad b_1 = 0 \end{cases}$	$U(t) = (u^{-(b_1+b_2)} - (b_1+b_2)t)^{-1/(b_1+b_2)}$ $V(t) = \begin{cases} v + \frac{(u^{-(b_1+b_2)} - (b_1+b_2)t)^{1-\frac{b_2}{b_1+b_2}} - u^{b_2}}{b_1}, \\ \qquad b_1 \neq 0 \\ v + \ln\left(\frac{u^{-(b_1+b_2)} - (b_1+b_2)t}{u^{-(b_1+b_2)}}\right), \quad b_1 = 0 \end{cases}$	$U(t) = (u^{-(b_1+b_2)} - (b_1+b_2)t)^{-1/(b_1+b_2)}$ $V(t) = \begin{cases} \left[v^{-a_2} + \frac{a_2}{b_1}\left((u^{-(b_1+b_2)} - (b_1+b_2)t)^{b_1/(b_1+b_2)} - u^{-b_1}\right)\right]^{-1/a_2}, \quad b_1 \neq 0, \\ \left[v^{-a_2} - a_2 \ln\left(\frac{(u^{-(b_1+b_2)} - (b_1+b_2)t)^{-1/(b_1+b_2)}}{u}\right)\right]^{-1/a_2}, \quad b_1 = 0. \end{cases}$

This can be combine with Propositions 2.1 and 1.1 one

$$U(t)^{b_2+b_1+b_0} V(t)^{a_2+a_0} = \sum_{n \geq 0} \left(\sum_{k=0}^{n} T(n,k) u^{b_2 n + b_1 k + b_0 + b_1 + b_2} v^{a_2 n + a_1 k + a_0 + a_2} \right) \frac{t^n}{n!}.$$

For instance, if we take the case where $a_2, b_1 + b_2 \notin \{0, -1\}$, $b_1 \neq 0$ for $u = 1$ and $y = v^{a_1}, x = y^{a_2/a_1} t$,:

$$\begin{aligned} &(1 - (b_1 + b_2)t\, y)^{-(b_1+b_2+b_0)/(b_1+b_2)} \\ &\quad \cdot \left[y + \frac{a_2}{b_1} \left((1 - (b_1 + b_2)t\, y)^{b_1/(b_1+b_2)} - 1 \right) \right]^{-(a_2+a_0)/a_2} \\ &= y^{(a_0+a_2)/a_1} \sum_{n \geq 0} \frac{t^n}{n!} \sum_{k=0}^{n} T(n,k) y^k. \end{aligned}$$

Proposition 3.3. *If $a_2, b_1 + b_2 \notin \{0, -1\}$, $b_1 \neq 0$, then*

$$\begin{aligned} \sum_{n \geq 0} \sum_{k=0}^{n} T(n,k)\, y^k \frac{t^n}{n!} &= y^{-\frac{a_0+a_2}{a_1}} \left(1 - (b_1 + b_2)t\, y^{-\frac{a_2}{a_1}} \right)^{-\frac{b_0+b_1+b_2}{b_1+b_2}} \\ &\quad \times \left[y + \frac{a_2}{b_1} \left((1 - (b_1 + b_2)t\, y)^{\frac{b_1}{b_1+b_2}} - 1 \right) \right]^{-\frac{a_0+a_2}{a_2}} \\ &= (1 - (b_1 + b_2)t\, y)^{-\frac{b_0+b_1+b_2}{b_1+b_2}} \left[1 + \frac{a_2}{b_1 y} \left((1 - (b_1 + b_2)t\, y)^{\frac{b_1}{b_1+b_2}} - 1 \right) \right]^{-\frac{a_0+a_2}{a_2}}. \end{aligned}$$

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